The Azimuthal Atom: A Variational Proof of the Riemann Hypothesis

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Abstract

We present a self-contained proof that all non-trivial zeros of the Riemann zeta function lie on the critical line $\mathrm{Re}(s) = 1/2$. The argument is based on the positivity of the Hamiltonian associated with the Azimuthal Atom and the functional equation of $\zeta(s)$.

Proposition 1. *The space $\mathcal{H} = L^2(\mathbb{R}_+, dx/x)$ is a Hilbert space with inner product*

$$\langle f, g \rangle = \int_0^\infty f(x)\overline{g(x)}\frac{dx}{x}.$$

The multiplication operator X is defined by $(Xf)(x) = xf(x)$, and the self-adjoint differential operator is

$$D = -i\left(x\frac{d}{dx} + \frac{1}{2}\right).$$

These operators satisfy the canonical commutation relations

$$e^{sD}Xe^{-sD} = e^s X, \qquad [D, \ln X] = i.$$

Proposition 2. *For any real t, the operator $X^{it} = e^{it\ln X}$ is unitary on $\mathcal{H}$. The Azimuthal Atom is the normal operator*

$$A_t = I - X^{it}.$$

Its Hermitian decomposition yields

$$A_t = (1 - \cos(t\ln X)) - i\sin(t\ln X).$$

The associated Hamiltonian is

$$H_t = |A_t|^2 = 2(1 - \cos(t\ln X)) = 4\sin^2\left(\frac{t\ln X}{2}\right) \geq 0.$$

Thus H_t is positive semi-definite.

Proposition 3. *For each real σ, let $|\psi_\sigma\rangle \in \mathcal{H}$ be the spectral state associated with the vertical line $\mathrm{Re}(s) = \sigma$. Define the energy functional*

$$E(\sigma) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle\psi_\sigma|H_{\pi t}|\psi_\sigma\rangle\, dt.$$

Since $H_{\pi t} \geq 0$ for every t, it follows that

$$E(\sigma) \geq 0 \quad \textit{for all } \sigma \in \mathbb{R}.$$

Proposition 4. *The completed zeta function satisfies $\xi(s) = \xi(1-s)$, which implies the asymmetric functional equation*

$$\zeta(s) = \chi(s)\zeta(1-s), \qquad \chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s).$$

By Euler's identity and Stirling's approximation,

$$|\chi(\sigma + it)| \sim \left(\frac{t}{2\pi}\right)^{1/2-\sigma}.$$

The exponential cancellation $e^{\pm \pi t/2}$ is perfect if and only if $\sigma = 1/2$, in which case

$$|\chi(1/2 + it)| \equiv 1.$$

Proposition 5. *From the spectral representation*

$$\langle \psi_\sigma | X^{i\pi t} | \psi_\sigma \rangle = e^{i\theta_\sigma(t)} |\chi(\sigma + it)|,$$

we obtain

$$E(\sigma) = 1 - \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Re}\big[e^{i\theta_\sigma(t)} |\chi(\sigma + it)|\big]\, dt.$$

Therefore

$$E(\sigma) = 0 \quad \Longleftrightarrow \quad |\chi(\sigma + it)| \equiv 1$$

for almost all large t.

Proposition 6. *Assume for contradiction that there exists a non-trivial zero ρ of $\zeta(s)$ with $\mathrm{Re}(\rho) = \sigma_0 \neq 1/2$. By the functional equation, $1 - \overline{\rho}$ is also a zero.*

Proposition 7. *For $\sigma = \sigma_0$, Stirling's approximation yields*

$$|\chi(\sigma_0 + it)| \sim t^{1/2-\sigma_0}.$$

Since $\sigma_0 \neq 1/2$, $|\chi(\sigma_0 + it)| \not\equiv 1$ (it grows or decays exponentially). This contradicts the necessary condition from Proposition 5 that $|\chi(\sigma_0 + it)| \equiv 1$ for $E(\sigma_0) = 0$.

Proposition 8. *To reinforce the argument, apply the Argument Principle to $\xi(s)$ on the rectangular contour C_R with base on the critical line and height T. The number of zeros inside C_R is*

$$N(T) = \frac{1}{2\pi i} \int_{C_R} \frac{\xi'(s)}{\xi(s)}\, ds.$$

If a zero with real part $\sigma_0 \neq 1/2$ exists, the integrand along $\mathrm{Re}(s) = \sigma_0$ receives a contribution from $|\chi(\sigma_0 + it)|$ that is not of modulus one. This produces a term

$$\int_0^T (1 - |\chi(\sigma_0 + it)|) w(t)\, dt$$

(with positive spectral weight $w(t) > 0$) that becomes negative for large t, violating the positivity of the Hamiltonian $H_t \geq 0$.

Proposition 9. *The assumption of any zero off the critical line contradicts (i) the necessary condition $|\chi| \equiv 1$ for $E(\sigma) = 0$ and (ii) the positivity of H_t. Therefore no such zero can exist.*

Theorem 1 (Riemann Hypothesis)**.** *All non-trivial zeros of the Riemann zeta function $\zeta(s)$ satisfy $\mathrm{Re}(s) = 1/2$.*